

# CLUB GUESSING FOR DUMMIES

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**ABSTRACT.** We give a direct, detailed and relatively short proof of Shelah's theorem on club guessing sequences on  $S_\mu^{\mu^+}$  (for any regular, uncountable cardinal  $\mu$ ).

## 1. INTRODUCTION

The aim of this paper is to give an easy proof of a less known club guessing theorem of Shelah. We will use the following notations. For a cardinal  $\lambda$  and a regular cardinal  $\mu$  let  $S_\mu^\lambda$  denote the ordinals in  $\lambda$  with cofinality  $\mu$ . For  $S \subseteq S_\mu^\lambda$  an *S-club sequence* is a sequence  $\underline{C} = \langle C_\delta : \delta \in S \rangle$  such that  $C_\delta \subseteq \delta$  is a club in  $\delta$  of order type  $\mu$ . One of the basic results in Shelah's club guessing theory is the following.

**Theorem 1.1** ([2, Claim 2.3]). *Let  $\lambda$  be a cardinal such that  $cf(\lambda) \geq \mu^{++}$  for some regular  $\mu$  and let  $S \subseteq S_\mu^\lambda$  stationary. Then there is an S-club sequence  $\underline{C} = \langle C_\delta : \delta \in S \rangle$  such that for every club  $E \subseteq \lambda$  there is  $\delta \in S$  (equivalently, stationary many) such that  $C_\delta \subseteq E$ .*

Actually Shelah in [2] proves more, but a detailed proof of Theorem 1.1 can be found in [1, Theorem 2.17].

What can be said about guessing clubs on  $S_\mu^{\mu^+}$ ? It is known that in ZFC there are no such strong guessing sequences generally but some approximation can be done in this case either.

## 2. CLUB GUESSING ON $S_\mu^{\mu^+}$

**Theorem 2.1** ([3, Claim 3.3]). *Let  $\lambda$  be a cardinal such that  $\lambda = \mu^+$  for some uncountable, regular  $\mu$  and  $S \subseteq S_\mu^\lambda$  stationary. Then there is an S-club sequence  $\underline{C} = \langle C_\delta : \delta \in S \rangle$  such that  $C_\delta = \{\alpha_\zeta^\delta : \zeta < \mu\} \subseteq \delta$  and for every club  $E \subseteq \lambda$  there is  $\delta \in S$  (equivalently, stationary many) such that:*

$$\{\zeta < \mu : \alpha_{\zeta+1}^\delta \in E\} \text{ is stationary.}$$

*Proof.* Let us prove something easier first.

**Claim 2.2.** *There is an S-club sequence  $\underline{C} = \langle C_\delta : \delta \in S \rangle$ , such that  $C_\delta = \{\alpha_\zeta^\delta : \zeta < \mu\}$  and for every club  $E \subseteq \lambda$  there is  $\delta \in S$  (equivalently, stationary many) such that*

$$\{\zeta < \mu : (\alpha_\zeta^\delta, \alpha_{\zeta+1}^\delta) \cap E \neq \emptyset\} \text{ is stationary.}$$

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*Proof.* Suppose on the contrary that this does not hold. Take any  $S$ -club sequence  $\underline{C}_0 = \langle C_\delta^0 : \delta \in S \rangle$ . This does not satisfy the above claim, thus there is some club  $E_0 \subseteq \lambda$  such that for every  $\delta \in S$  there is a club  $e_\delta^0 \subseteq \delta$  such that

$$\alpha \in e_\delta^0 \cap C_\delta^0 \Rightarrow E_0 \cap (\alpha, \min(C_\delta^0 \setminus \alpha + 1)) = \emptyset$$

Let  $C_\delta^1 = C_\delta^0 \cap e_\delta^0$  for  $\delta \in S$  and  $\underline{C}_1 = \langle C_\delta^1 : \delta \in S \rangle$ . If we defined  $\underline{C}_n$  for some  $n \in \omega$  then  $\underline{C}_n$  does not guess well each club, thus there is some club  $E_n \subseteq \lambda$  such that for every  $\delta \in S$  there is a club  $e_\delta^n \subseteq \delta$  such that

$$\alpha \in e_\delta^n \cap C_\delta^n \Rightarrow E_n \cap (\alpha, \min(C_\delta^n \setminus \alpha + 1)) = \emptyset$$

Let  $C_\delta^{n+1} = C_\delta^n \cap e_\delta^n$  for  $\delta \in S$  and  $\underline{C}_{n+1} = \langle C_\delta^{n+1} : \delta \in S \rangle$ .

Since  $cf(\delta) = \mu > \omega$  for  $\delta \in S$ ,  $C_\delta = \bigcap \{C_\delta^n : n \in \omega\}$  is a club in  $\delta$ . Let  $E = \bigcap \{E_n : n \in \omega\}$  and pick  $\delta \in E \cap S$  such that  $tp(\delta \cap E) > \mu$ . Thus there is some  $\xi < \delta$  such that  $\xi \in E \setminus C_\delta$  and  $\min C_\delta < \xi$ . Since the sequence  $\langle C_\delta^n : n \in \omega \rangle$  is decreasing,  $\sup(C_\delta^n \cap \xi)$  is decreasing either thus there is some  $m \in \omega$  such that  $\alpha = \sup(C_\delta^m \cap \xi) < \xi$  for  $n \geq m$ . Since  $\alpha = \sup(C_\delta^{n+1} \cap \xi) \in C_\delta^{n+1} = e_\delta^n \cap C_\delta^n$  and  $\xi \in E_n$ ,  $\alpha < \xi < \min(C_\delta^n \setminus \alpha + 1)$ , this contradicts the fact that  $E_n \cap (\alpha, \min(C_\delta^n \setminus \alpha + 1)) = \emptyset$ .  $\square$

Let  $\langle C_\delta : \delta \in S \rangle$  be an  $S$ -club sequence given by Claim 2.2, with continuous enumeration  $C_\delta = \{\alpha_\zeta^\delta : \zeta < \mu\}$ . For any club  $E \subseteq \lambda$  and  $\delta \in S$  define the following partial function  $f_{\delta, E}$  on  $\mu$ :

$$f_{\delta, E}(\zeta) = \sup(E \cap (\alpha_\zeta^\delta, \alpha_{\zeta+1}^\delta))$$

Claim 2.2 states that for any club  $E \subseteq \lambda$  there is stationary many  $\delta \in S$  such that  $\text{dom} f_{\delta, E}$  is stationary in  $\mu$ .

**Claim 2.3.** *There is a club  $E_0 \subseteq \lambda$  such that for each club  $E \subseteq E_0$  there is  $\delta \in S$  (equivalently, stationary many) such that*

$$\{\zeta < \mu : \zeta \in \text{dom} f_{\delta, E} \text{ and } f_{\delta, E}(\zeta) = f_{\delta, E_0}(\zeta)\} \text{ is stationary.}$$

*Proof.* Suppose on the contrary that this does not hold. Thus  $E_0 = \lambda$  is not good, so there is some club  $E_1 \subseteq E_0$  such that for every  $\delta \in S$  there is some club  $d_\delta^1 \subseteq \mu$  such that

$$\zeta \in d_\delta^1 \Rightarrow \zeta \notin \text{dom} f_{\delta, E_1} \text{ or } f_{\delta, E_1}(\zeta) < f_{\delta, E_0}(\zeta).$$

If we have the club  $E_n$  for some  $n \in \omega$ , then there is some club  $E_{n+1} \subseteq E_n$  such that for every  $\delta \in S$  there is some club  $d_\delta^{n+1} \subseteq \mu$  such that

$$\zeta \in d_\delta^{n+1} \Rightarrow \zeta \notin \text{dom} f_{\delta, E_{n+1}} \text{ or } f_{\delta, E_{n+1}}(\zeta) < f_{\delta, E_n}(\zeta).$$

Let  $E = \bigcap \{E_n : n \in \omega\}$ , then  $E$  is a club. There is stationary many  $\delta \in S$  such that  $\text{dom} f_{\delta, E}$  is stationary, let  $\delta \in E$  such that  $\text{dom} f_{\delta, E}$  is stationary. Since for  $n \in \omega : E \subseteq E_n$  thus  $\text{dom} f_{\delta, E} \subseteq \text{dom} f_{\delta, E_n}$ . Let  $d = \bigcap \{d_\delta^n : n \in \omega\}$ , then  $d \subseteq \mu$  is a club. Thus there is some  $\zeta \in d \cap \text{dom} f_{\delta, E}$ , thus  $\zeta \in \text{dom} f_{\delta, E_n}$  for each  $n \in \omega$ . But then by the definition of the sets  $d_\delta^n$  we have an infinite decreasing sequence of ordinals:

$$f_{\delta, E_0}(\zeta) > f_{\delta, E_1}(\zeta) > \dots > f_{\delta, E_n}(\zeta) > f_{\delta, E_{n+1}}(\zeta) > \dots$$

which is a contradiction.  $\square$

With the aid of this club  $E_0$  we modify the sequence  $\langle C_\delta : \delta \in S \rangle$ . Let  $S_0 = \{\delta \in S : \text{dom} f_{\delta, E_0} \text{ is stationary in } \mu\} \subseteq S$ , then  $S_0$  is stationary. Let  $\tilde{C}_\delta = C_\delta$  for  $\delta \in S \setminus S_0$ . Let  $\tilde{C}_\delta = C_\delta \cup \{f_{\delta, E_0}(\zeta) : \zeta \in \text{dom} f_{\delta, E_0}\}$ , clearly  $\tilde{C}_\delta$  is a club in  $\delta$ . We claim that the sequence  $\langle \tilde{C}_\delta : \delta \in S \rangle$  has the desired property. Let  $D \subseteq \lambda$  be any club, let  $E = D \cap E_0 \subseteq E_0$ . Then by Claim 2.3 for stationary many  $\delta \in S$   $\{\zeta < \mu : \zeta \in \text{dom} f_{\delta, E} \text{ and } f_{\delta, E}(\zeta) = f_{\delta, E_0}(\zeta)\}$  is stationary. For such a  $\zeta < \mu$ , the successor of  $\alpha_\zeta^\delta$  in  $\tilde{C}_\delta$  is  $f_{\delta, E_0}(\zeta)$  and  $f_{\delta, E_0}(\zeta) \in D$ , since  $f_{\delta, E_0}(\zeta) = f_{\delta, E}(\zeta) \in E = E_0 \cap D$ .  $\square$

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